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CONE BUNDLES

BY

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ABSTRACT. A theory of normal bundles for locally knotted codimension two embeddings of PL manifolds is developed. The classifying space for this theory is Cappell and Shaneson's space BRN_2.

Cone bundles are a generalization of blockbundles [9], allowing local knotting of the base in the total space. They are a type of "mockbundle" [8], [2], closely related to the theory of stratified polyhedra [12], and designed to provide a simple foundation for Cappell and Shaneson's theory of singularities of PL embeddings [3], [4]. A similar definition has been given by Matumoto and Matsumoto [6].

§1 contains the basic definitions. A classifying space for cone bundles is constructed in §2. §3 contains a proof that the total space of a cone bundle over a manifold is a manifold. Finally, cone bundles are related to the topology of stratified polyhedra in §4.

The geometric idea for cone bundles comes from my paper [7] on cone complexes. I thank Sylvain Cappell for encouraging me to develop this idea.

I will work in the category of polyhedra and piecewise linear maps [11]. In particular, all manifolds and homeomorphisms will be piecewise linear.

1. **Thickenings.** Let M be a compact n-manifold. A codimension q *thickening* of M is a compact $(n+q)$-manifold W containing M as a subpolyhedron, such that W collapses to M. Furthermore, $\partial W \cap M = \partial M$, and there is a collar of ∂W in W which restricts to a collar of ∂M in M. (This is called a "very proper" thickening in [4].)

The thickenings V and W of M are *equivalent* if there is a homeomorphism between V and W which is the identity on M.

If $q > 2$, a codimension q thickening W of M is an "abstract regular neighborhood" of M [9, p. 14], since M is locally flat in W. But if $q = 2$, M can be locally knotted in W. M can be locally knotted in a codimension one thickening if and only if the PL Schoenflies conjecture is false.

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Let K be a (PL) cell complex. A *q-cone bundle* ξ/K consists of a polyhedron $E(\xi)$ containing $|K|$ such that

(i) For each p-cell $\sigma_i \in K$ there is a $(p+q)$-ball $\beta_i \subset E(\xi)$, containing σ_i, such that (β_i, σ_i) is homeomorphic with the cone on a sphere pair. (N.B. this sphere pair may be nonlocally flat or knotted.) β_i is called the *block* over σ_i.

(ii) $E(\xi)$ is the union of the blocks β_i.

(iii) The interiors of the blocks are disjoint.

(iv) $\beta_i \cap \beta_j$ is the union of the blocks over the cells contained in $\sigma_i \cap \sigma_j$.

By the Zeeman unknotting theorem, a q-cone bundle is a blockbundle [9] if $q > 2$, i.e., (β_i, σ_i) is homeomorphic with a standard sphere pair for each $\sigma_i \in K$. The following results show that cone bundles bear essentially the same relation to thickenings that blockbundles bear to abstract regular neighborhoods.

LEMMA 1. *Let N be a manifold containing the manifold M as a subpolyhedron. Suppose that $\partial N \cap M = \partial M$ and $(\partial N, \partial M)$ is collared in (N, M). Then there is a cone bundle ξ/K with $|K| = M$ such that $E(\xi)$ is a regular neighborhood of M in N.*

PROOF. ξ is constructed in the same manner as a normal blockbundle for a locally flat submanifold, using dual cells. (The usual construction must be slightly modified near the boundary; cf. [7, p. 284].)

LEMMA 2. *If ξ/K is a cone bundle, and $|K|$ is a compact manifold, then $E(\xi)$ is a thickening of K.*

PROOF. It is easy to see that $E(\xi)$ collapses to $|K|$, since $E(\xi)$ can be triangulated as a stellar neighborhood of $|K|$, by induction on the dimension of ξ (cf. [7, p. 274]). The fact that $E(\xi)$ is a *manifold* will be proved in §3.

LEMMA 3. *If W is a thickening of the compact manifold M, there is a cone bundle ξ/K, $|K| = M$, such that the thickening $E(\xi)$ of M is equivalent to W.*

PROOF. This follows from Lemma 1 and the uniqueness of regular neighborhoods [11, p. 33].

REMARKS. (1) If ξ/K and η/L are cone bundles with $|K| = |L| = M$, and the thickenings $E(\xi)$ and $E(\eta)$ of M are equivalent, one might expect (by analogy with blockbundles) that ξ and η have isomorphic "subdivisions". However, this is not true–one has to introduce the weaker relation of *concordance* (§2) in order to get a bijection between classes of bundles and classes of thickenings.

(2) In Matumoto and Matsumoto's definition of "RN_2-bundles" [6], condition (i) in the definition of a 2-cone bundle is weakened to the condition that (β_i, σ_i) is an arbitrary ball pair. Lemma 2 is not true for their bundles, since the total space need not collapse to the base space.

(3) A 2-cone bundle ξ/K has a canonical "Noguchi characteristic class" $\mathfrak{n} \in H^2(K; \gamma)$ (twisted coefficients), where γ is the Fox-Milnor cobordism group (cf. [4]). $\mathfrak{n}$ is represented by the cocycle which assigns to each 2-cell $\sigma \in K$, the cobordism class of the knot $(\partial\beta, \partial\sigma)$, where β is the block over σ. Thus $\mathfrak{n}$ is the primary obstruction to making ξ a blockbundle. (The analogous higher obstructions are not defined *a priori* since $\partial\sigma_i$ is not necessarily locally flat in $\partial\beta_i$ if $\dim \sigma_i > 2$.)

2. **A classifying space.** The following definitions come from [9].

If ξ/K is a cone bundle and L is a subcomplex of K, the *restriction* $\xi|L$ is defined by putting $\beta_i(\xi|L) = \beta_i(\xi)$ for each $\sigma_i \in L$.

The cone bundles ξ_0, ξ_1/K are *isomorphic* if there is a homeomorphism h: $E(\xi_0) \to E(\xi_1)$ such that h is the identity on $|K|$ and $h(\beta_i(\xi_0)) = \beta_i(\xi_1)$ for each $\sigma_i \in K$.

The cone bundles ξ_0, ξ_1/K are *concordant* if there is a cone bundle $\eta/(K \times I)$ such that $\eta|(K \times \{i\})$ is isomorphic with ξ_i, $i = 0, 1$. Here $I = [0, 1]$ and $K \times I$ is the usual product complex. (Two blockbundles are concordant if and only if they are isomorphic [9, p. 6]. This is not true for 2-cone bundles.)

We will construct a classifying space for concordance classes of 2-cone bundles analogous to the classifying space $B\widetilde{PL}_q$ for q-blockbundles. (The same construction also works for 1-cone bundles.)

Let $\mathcal{K}(K)$ be the set of concordance classes of 2-cone bundles over K. $\mathcal{K}$ is a contravariant functor from the category with objects PL cell complexes and morphisms generated by isomorphisms and inclusions of subcomplexes, to the category of (based) sets. (The base point of $\mathcal{K}(K)$ is the class of the trivial bundle over K.)

THEOREM 1. *$\mathcal{K}$ has a unique extension to the category of CW complexes and homotopy classes of maps.*

PROOF. This is a corollary of the "mockbundle" recipe for homotopy functors [2, I]. It is clear that cone bundles can be glued (axiom G [2, p. 15]), so we only have to verify the extension axiom (E, [2, p. 15]). That is, if e: $K_0 \to K$ is an elementary expansion, and ξ_0/K_0 is a cone bundle, we must construct a cone bundle ξ/K such that $\xi|K_0 = \xi_0$. We follow [2, p. 21]. $K = K_0 \cup \{\sigma, \tau\}$, where σ is a principal cell of K and τ is a free face of σ. Let J be the subcomplex of $\partial\sigma$ consisting of all the faces of σ except τ. Now $|J|$ is a ball, and $E(\xi_0|J)$ is a thickening of $|J|$ by Lemma 2, so $E(\xi_0|J)$ is a ball. Let (B, C) denote the ball pair $(E(\xi_0|J), |J|)$. Identifying (σ, C, τ) with $(C \times I, C \times \{0\}, (\partial C \times I) \cup (C \times \{1\}))$, we define the extension ξ/K as follows. The block of ξ over σ is the cone on the boundary of the ball $B \times I$, and the block over τ is the cone on the boundary of the ball

$$\mathrm{cl}\left[\partial(B \times I) \setminus (B \cup (B^{\bullet} \times I))\right],$$

where $B^{\bullet} = \mathrm{cl}[\partial B \setminus \bigcup \beta_i(\xi_0)]$, union over all i such that $\sigma_i \subset \partial C$.

THEOREM 2. *$\mathcal{H}$ is a representable functor.*

PROOF. Let G be the (based) Δ-set whose k-simplexes are 2-cone bundles over Δ^k (the standard k-simplex) which are embedded blockwise in $\Delta^k \times R^\infty$, and let γ be the canonical cone bundle on G (cf. [8, p. 131] and [2, p. 37]). G is a Kan Δ-set by the extension axiom (see the proof of Theorem 1) and general position. It follows that if $\mathcal{G}$ is the realization of G, pulling back the class of γ induces a bijection

$$\mathcal{H}(X) \cong [X, \mathcal{G}]$$

for all CW complexes X, where [,] denotes homotopy classes of maps (cf. [10, §6] and [9, §2]).

Theorems 1 and 2 are also true for Matumoto and Matsumoto's RN_2-bundles, by the same proofs. As they have pointed out to me, any RN_2-bundle is concordant to a cone bundle by the Alexander trick, so the corresponding homotopy functors are the same.

The thickenings W_0 and W_1 of the n-manifold M are *concordant* if there is a thickening Q of $M \times I$ such that W_i is a regular neighborhood of $M \times \{i\}$ rel $\partial M \times \{i\}$ in ∂Q, $i = 0, 1$ (cf. [4]).

THEOREM 3. *If M is a compact n-manifold, $\xi \mapsto E(\xi)$ induces a bijection between $\mathcal{H}(M)$ and concordance classes of codimension 2 thickenings of M.*

PROOF. Every thickening of M is in fact *equivalent* to $E(\xi)$ for some ξ over M, by Lemma 3. On the other hand, given ξ_0 and ξ_1, and a concordance Q between $E(\xi_0)$ and $E(\xi_1)$, a concordance between ξ_0 and ξ_1 can be constructed as a regular neighborhood of $M \times I$ in Q, by the relative version of Lemma 1.

It follows that the classifying space $\mathcal{G}$ is (canonically homotopy equivalent with) Cappell and Shaneson's classifying space BRN_2 [3], [4]. In the same way, "oriented" 2-cone bundles are classified by $BSRN_2$, and 2-cone bundles which are blockbundles on the $(k-1)$-skeleton are classified by $BRN_{2,k}$.

REMARKS. (1) The Noguchi obstruction $\mathfrak{n}$ can be viewed as a natural transformation from $\mathcal{H}(\cdot)$ to $H^2(\cdot;\, \gamma)$, since $\mathfrak{n}(\xi)$ depends only on the concordance class of ξ. Furthermore, $\mathfrak{n}(\xi) = 0$ if and only if ξ is concordant to a cone bundle which is a blockbundle on the 2-skeleton (cf. [4, §3]).

(2) In [4], Cappell and Shaneson completely determine the homotopy type of $BSRN_2$. An interesting problem is to give a geometric description of the resulting H-space structure on $BSRN_2$.

3. **Collared complexes.** A *collared complex* $\mathcal{C}$ on a polyhedron $X = |\mathcal{C}|$ is a locally finite covering of X by compact subpolyhedra, together with a subpolyhedron $\delta\alpha$ of each element α of $\mathcal{C}$ such that

(i) for each $\alpha \in \mathcal{C}$, $\delta\alpha$ is a union of elements of $\mathcal{C}$,

(ii) if α and β are distinct elements of $\mathcal{C}$, $\alpha^\circ \cap \beta^\circ$ is empty, where $\alpha^\circ = \alpha \setminus \delta\alpha$,

(iii) $\delta\alpha$ is collared in α for each $\alpha \in \mathcal{C}$. ((i) and (ii) imply that $\alpha \cap \beta$ is a union of elements of $\mathcal{C}$.)

Collared complexes are Akin's "general complexes" [1]. Examples of collared complexes are cell complexes, manifold complexes [5], and cone complexes [7].

The usefulness of collared complexes comes from the following proposition, derived from the proof of a lemma of Cohen and Sullivan [5, p. 142]. (See also [2, p. 21] and [7, p. 278].)

If $\mathcal{C}$ is a collared complex and $\alpha \in \mathcal{C}$, let $L(\alpha)$ be the geometric realization of the nerve of the finite partially ordered set $\{\beta \in \mathcal{C}, \alpha < \beta\}$, where $\alpha < \beta$ means $\alpha \subset \delta\beta$.

PROPOSITION 1. *If $\mathcal{C}$ is a collared complex on X, $\alpha \in \mathcal{C}$, and $x \in \alpha^\circ$, then*

$$lk(x; X, \alpha) \cong (L(\alpha) * lk(x; \alpha), lk(x; \alpha)),$$

where lk denotes the link, and $*$ denotes the join.

PROOF. Use induction on the "depth" of α, i.e. the length of a maximal chain $\alpha < \alpha_1 < \cdots < \alpha_n$ in $\mathcal{C}$.

With this proposition, we can prove Lemma 2, by induction on the dimension of the base. Let ξ/K be a cone bundle, with $|K|$ a manifold (with boundary). If $\sigma_i \in K$, and β_i is the block of ξ over σ_i, let $\delta\beta_i = E(\xi|\partial\sigma_i)$. By induction hypothesis, $\delta\beta_i$ is a codimension 0 submanifold of $\partial\beta_i$, so $\delta\beta_i$ is collared in β_i. Thus the set of blocks of ξ forms a collared complex $\mathcal{C}$ on $E(\xi)$. The map $\beta_i \mapsto \sigma_i$ is an incidence preserving bijection between $\mathcal{C}$ and K. Therefore $L(\beta_i) = L(\sigma_i)$ for all $\sigma_i \in K$. Thus the proposition implies $E(\xi)$ is a manifold (with boundary), since each block β_i is a manifold and $|K|$ is a manifold.

4. **Geometry of codimension 2 thickenings.** Let W be a codimension 2 thickening of the compact n-manifold M. If $x \in M$, the *intrinsic dimension* $d(x; W, M)$ is the smallest integer k such that x is in the k-skeleton of every (PL) cell complex on W which has M as a subcomplex. The kth *intrinsic stratum* S_k of M in W is

$$\{x \in M \setminus \partial M, d(x; W, M) = k\} \cup \{x \in \partial M, d(x; \partial W, \partial M) = k - 1\}.$$

(Cf. [12, p. 13]. Recall that $(\partial W, \partial M)$ is collared in (W, M).) S_k is a k-dimensional submanifold of M, and $\mathrm{cl}(S_k) = \bigcup_{j<k} S_j$. S_n is the set of

locally flat points of M in W, and S_k can be thought of as the points at which the "degree of local knottedness" of M in W is $n - k$. Let $\mathcal{S} = \{S_k\}$ denote this *intrinsic stratification* of M in W.

By Lemma 3, we can assume that $W = E(\xi)$ for some 2-cone bundle ξ/K, $|K| = M$. Now for each block β of ξ, choose a cellular subdivision of the "rim" $\beta^{\bullet} = \mathrm{cl}(\partial\beta \setminus \delta\beta)$. These cells, together with the blocks themselves, form a cell complex $\mathcal{C}$ on W. Choose a cone structure for each cell of $\mathcal{C}$ so that σ_i is a subcone of β_i for each cell $\sigma_i \in K$. (N.B. K is not a subcomplex of $\mathcal{C}$.) Then the *dual* cone complex $\mathcal{C}^*$ on W [7] will have the complex K^* on M as a subcomplex. (Note that the cones of $\mathcal{C}^*$ are cells, but if $\alpha \in \mathcal{C}^*$ and $\alpha \cap \partial W \neq \emptyset$, the apex of α lies in ∂W.) It follows that $\mathrm{cl}(S_k)$ is a subcomplex of K^* for all k, i.e. *the cells of K are transverse to the intrinsic stratification of M in W.* (See [7, p. 287] for a discussion of transversality to a stratification.)

If K' is a subdivision of K, the cone bundle ξ'/K' is a *subdivision* of ξ/K if for each $\sigma_i \in K$, $\beta_i(\xi) = \bigcup \beta_i(\xi')$, where the union is taken over all blocks $\beta_j(\xi')$ over cells $\tau_j \in K'$ such that $\tau_j \subset \sigma_i$.

It is not hard to see that a subdivision ξ' of ξ will exist over the subdivision K' of K if and only if $(K')^*$ extends to a cell complex on $W = E(\xi)$ (for some cone structuring of K'). This is equivalent to the condition that K' be transverse to the intrinsic stratification $\mathcal{S}$. Therefore, ξ can be restricted to precisely those subpolyhedra of M which are transverse to $\mathcal{S}$.

Thus the fact that concordance classes of cone bundles can be "pulled back" is a consequence of the geometric fact that any subpolyhedron X of the manifold M can be moved transverse to $\mathcal{S}$. (In fact, Stone's transversality theorem [12] can be easily proved from the mockbundle viewpoint–cf. [7, p. 287].)

The following result is important in [4].

PROPOSITION 2. *Let W be a codimension 2 thickening of M, and let N be a locally flat codimension q submanifold of M, with $\partial M \cap N = \partial N$. Suppose that N is transverse to the intrinsic stratification $\mathcal{S}$ of M in W. Then there is a cone bundle ξ over M with $E(\xi) = W$, and a normal blockbundle ν of N in M such that $E(\nu)$ is transverse to $\mathcal{S}$, and $E(\xi|E(\nu))$ is a codimension q thickening of $E(\xi|N)$ equivalent to $E(q^*\nu)$, where q: $E(\xi|N) \to N$ is a homotopy inverse of the inclusion.*

PROOF. N is transverse to $\mathcal{S}$ implies there is a cone bundle η/L, $|L| = M$, with $E(\eta) = W$ and N a subcomplex of L. Let K be the canonical "full" subdivision of L constructed in [7, p. 276], and let ξ be a subdivision of η over K. (It is easy to construct ξ explicitly.) Then the union of the cells in K which meet N is a regular neighborhood of N, and so this neighborhood equals $E(\nu)$ for some blockbundle ν over N. $E(\nu)$ is transverse to $\mathcal{S}$ since it is a subcomplex of K. $E(\xi|E(\nu))$ is a manifold by Lemma 2, and it collapses to $E(\xi|N)$ since $E(\nu)$ collapses to N. Thus $E(\xi|E(\nu))$ is a thickening of $E(\xi|N)$. $E(\xi|N)$ is locally flat in $E(\xi|E(\nu))$ by Proposition 1, since the given collared complexes on $E(\xi|E(\nu))$ and $E(\nu)$ are abstractly isomorphic. Thus $E(\xi|E(\nu)) \supset E(\xi|N)$ is equivalent to $E(q^*\nu) \supset E(\xi|N)$ by the uniqueness of regular neighborhoods.

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